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One-Parameter Indecomposable and Irreducible Representations of gl(2 | 1) Superalgebra

Yong-Qing Chen[1]

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Using inhomogeneous boson–fermion realization, one-parameter indecomposable and irreducible representations of the gl(2 | 1) superalgebra are studied on subspace and quotient spaces of the universal enveloping algebra of Heisenberg–weyl superalgebra. All the finite-dimensional irreducible representations of one-parameter of the gl(2 | 1) superalgebra are naturally obtained as special cases. The parameter has relation to the Hubbard interaction parameter U in the Hubbard model for correlated electrons.

1. INTRODUCTION

A series of models of correlated electrons on a lattice and exactly solvable in one dimension and supersymmetric, such as Hubbard and extended Hubbard models and *t-J* model (Essler and Korepin, 1992, 1994; Sakar, 1990, 1991), EKS model (Essler *et al.*, 1992, 1993), BGLZ model (Brachen *et al.*, 1995), has been extensively studied because of their promising role in theoretical condensed-matter physics and possibly in high-T_c superconductivity. Those models contain one symmetry-preserving free real parameter, which is the Hubbard interaction parameter U. The supersymmetry algebra of BGLZ model for correlated electrons on the unrestricted 4^L-dimensional electronic Hilbert space $\otimes_{n=1}^{L} C^4$ is superalgebra gl(2 | 1). The indecomposable representations of Lie superalgebras have many important applications in description of unstable particle systems (Dirac, 1984). Therefore, it is very important to study the new one-parameter indecomposable and irreducible representations of the gl(2 | 1). In the present paper we shall be concerned with the gl(2 | 1) superalgebra. It is quit a valid approach to employ the inhomogeneous boson–fermion realization of Lie superalgebras to study their indecomposable representations (Gruber *et al.*, 1983, 1984; Lenczewski and Gruber, 1986; Sun, 1987). In the present paper we shall study one-parameter indecomposable representations of the gl(2 | 1) superalgebra on the universal enveloping

[1] Shenzhen Institute of Education, Shenzhen 518029, People's Republic of China; e-mail: cyqldv@21cn.com

algebra of Heisenberg–Weyl superalgebras, and on their subspaces and quotient spaces using the inhomogeneous boson–fermion realizations of this superalgebra. All the finite-dimensional one-parameter irreducible representations of the gl(2 | 1) are naturally obtained as special cases on the subspaces of generalized Fock space.

2. ONE-PARAMETER INDECOMPOSABLE REPRESENTATION OF THE gl(2 | 1)

In accordance with Chen (2000) the generators of the gl(2 | 1) superalgebra read as follows:

$$\{Q_3, Q_+, Q_-, B, M \in \mathrm{gl}(2 \mid 1)_{\bar{0}} \mid V_+, V_-, W_+, W_- \in \mathrm{gl}(2 \mid 1)_{\bar{1}}\} \tag{1}$$

and satisfy the following commutation and anticommutation relations:

$$\begin{aligned}
&[Q_3, Q_\pm] = \pm 2Q_\pm, \quad [M, Q_\pm] = \mp Q_\pm, \quad [Q_+, Q_-] = Q_3,\\
&[B, Q_\pm] = [B, Q_3] = [B, M] = [M, Q_3] = 0,\\
&[Q_3, V_\pm] = \pm V_\pm, \quad [Q_3, W_\pm] = \pm W_\pm, \quad [B, V_\pm] = -V_\pm, \quad [B, W_\pm] = W_\pm,\\
&[Q_\pm, V_\mp] = V_\pm, \quad [Q_\pm, W_\mp] = -W_\pm, \quad [Q_\pm, V_\pm] = 0, \quad [Q_\pm, W_\pm] = 0,\\
&[M, V_+] = -V_+, \quad [M, W_-] = W_-, \quad [M, V_-] = [M, W_+] = 0\\
&\{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = \{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0,\\
&\{V_\pm, W_\pm\} = Q_\pm, \quad \{V_+, W_-\} = Q_3 + M, \quad \{V_-, W_+\} = M
\end{aligned} \tag{2}$$

In terms of one pair of boson operators and two pairs of fermion operators the inhomogeneous boson–fermion realization of the gl(2 | 1) may be represented as follows:

$$\begin{aligned}
Q_3 &= -n + 2b^+b + a_1^+a_1 + a_2^+a_2,\\
B &= (\alpha+1)n - a_1^+a_1 + a_2^+a_2\\
M &= (\alpha+1)n - b^+b - a_1^+a_1\\
Q_+ &= nb^+ - b^{+2}b - b^+a_1^+a_1 - b^+a_2^+a_2,\\
Q_- &= b\\
V_+ &= -\sqrt{\alpha}na_1^+ + \sqrt{\alpha+1}b^+a_2 + \sqrt{\alpha}a_1^+b^+b + \sqrt{\alpha}a_1^+a_2^+a_2\\
V_- &= \sqrt{\alpha}a_1^+b + \sqrt{\alpha+1}a_2\\
W_+ &= \sqrt{\alpha+1}na_2^+ + \sqrt{\alpha}b^+a_1 - \sqrt{\alpha+1}a_2^+b^+b - \sqrt{\alpha+1}a_2^+a_1^+a_1\\
W_- &= \sqrt{\alpha+1}a_2^+b - \sqrt{\alpha}a_1
\end{aligned} \tag{3}$$

Consider (1 + 2) states Heisenberg–Weyl superalgebra H:$\{b^+, b, a_1^+, a_1, a_2^+, a_2, E\}$, where E stands for the unit operator. According to the Poincare–Birckhoff–Witt theorem, we choose for its universal enveloping algebra Ω a basis

$$\left\{\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) = b^{+k} b^1 a_1^{+\alpha_1} a_1^{\beta_1} a_2^{+\alpha_2} a_2^{\beta_2} E^t \mid k, l, t \in Z^+, \alpha_1, \beta_1, \alpha_2, \beta_2 = 0,1\right\} \quad (4)$$

Each vector in the space of Ω is a linear combination of the basis with complex coefficients. Then, we consider an extension $\overline{\Omega}$ of the space Ω, in which each element is a linear combination of the basis whose coefficients are elements of the Grassmann algebra $\tilde{G}$.

The representation of the superalgebra H on the space of $\overline{\Omega}$ is defined as

$$\begin{aligned}
f(b^+)\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) &= \phi(k+1, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t)\\
f(b)\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) &= \phi(k, l+1, \alpha_1, \beta_1, \alpha_2, \beta_2, t)\\
&\quad + k\phi(k-1, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t+1)\\
f\left(a_1^+\right)\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) &= (1-\alpha_1)\phi(k, l, \alpha_1+1, \beta_1, \alpha_2, \beta_2, t)\\
f(a_1)\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) &= (-1)^{\alpha_1}\phi(k, l, \alpha_1, \beta_1+1, \alpha_2, \beta_2, t)\\
&\quad + \alpha_1\phi(k, l, \alpha_1-1, \beta_1, \alpha_2, \beta_2, t+1) \qquad (5)\\
f\left(a_2^+\right)\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) &= (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\phi(k, l, \alpha_1, \beta_1, \alpha_2+1, \beta_2, t)\\
f(a_2)\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) &= (-1)^{\alpha_1+\beta_1+\alpha_2}\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2+1, t)\\
&\quad + (-1)^{\alpha_1+\beta_1}\alpha_2\phi(k, l, \alpha_1, \beta_1, \alpha_2-1, \beta_2, t+1)
\end{aligned}$$

Now, we consider the quotient space V with the basis

$$V = (\overline{\Omega}/I)\colon \{\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2) = \phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, 0) \bmod I \mid k, l \in Z^+, \alpha_1, \beta_1, \alpha_2, \beta_2 = 0,1\} \quad (6)$$

corresponding to the two-side ideal I generated by the element E-1.

The representation (5) induces the new representation on the space of V

$$\begin{aligned}
f(b^+)\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2) &= \phi(k+1, l, \alpha_1, \beta_1, \alpha_2, \beta_2)\\
f(b)\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2) &= \phi(k, l+1, \alpha_1, \beta_1, \alpha_2, \beta_2)\\
&\quad + k\phi(k-1, l, \alpha_1, \beta_1, \alpha_2, \beta_2)\\
f\left(a_1^+\right)\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) &= (1-\alpha_1)\phi(k, l, \alpha_1+1, \beta_1, \alpha_2, \beta_2)\\
f(a_1)\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2) &= (-1)^{\alpha_1}\phi(k, l, \alpha_1, \beta_1+1, \alpha_2, \beta_2)\\
&\quad + \alpha_1\phi(k, l, \alpha_1-1, \beta_1, \alpha_2, \beta_2) \qquad (7)
\end{aligned}$$

$$f\left(a_2^+\right)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2) = (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\,\phi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2)$$
$$f(a_2)\,\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2) = (-1)^{\alpha_1+\beta_1+\alpha_2}\,\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1) + (-1)^{\alpha_1+\beta_1}\alpha_2\,\phi(k,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2)$$

Using the following relation

$$L\left(F\left(b^+, b, a_1^+, a_1, a_2^+, a_2\right)\right) = \tilde{F}\left(f(b^+),\ f(b),\ f\left(a_1^+\right),\ f(a_1),\ f\left(a_2^+\right),\ f(a_2)\right) \tag{8}$$

and the inhomogeneous boson–fermion realization (3), We obtain the representation L of the gl(2 | 1) on the space of V,

$$\begin{aligned}
&L(Q_3)\,\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad= (-n+2k+\alpha_1+\alpha_2)\,\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\qquad+ 2\,\phi(k+1,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\qquad+ (-1)^{\alpha_1}(1-\alpha_1)\,\phi(k,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)\\
&\qquad+ (-1)^{\alpha_2}(1-\alpha_2)\,\phi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1)\\
&L(B)\,\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad= [(\alpha+1)n-\alpha_1+\alpha_2]\,\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\qquad- (-1)^{\alpha_1}(1-\alpha_1)\,\phi(k,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)\\
&\qquad+ (-1)^{\alpha_2}(1-\alpha_2)\,\phi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1)\\
&L(M)\,\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad= [(\alpha+1)n-k-\alpha_1]\,\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\qquad- (-1)^{\alpha_1}(1-\alpha_1)\,\phi(k,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)\\
&\qquad- \phi(k+1,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&L(Q_+)\,\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad= (n-k-\alpha_1-\alpha_2)\,\phi(k+1,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\qquad- \phi(k+2,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\qquad- (-1)^{\alpha_1}(1-\alpha_1)\,\phi(k+1,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)\\
&\qquad- (-1)^{\alpha_2}(1-\alpha_2)\,\phi(k+1,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1)\\
&L(Q_-)\,\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2) = \phi(k,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\qquad+ k\,\phi(k-1,l,\alpha_1,\beta_1,\alpha_2,\beta_2)
\end{aligned} \tag{9}$$

$$
\begin{aligned}
&L(V_+)\,\phi\,(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad = -\sqrt{\alpha}\,n(1-\alpha_1)\,\phi\,(k,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&\qquad + (-1)^{\alpha_1+\beta_1+\alpha_2}\sqrt{\alpha+1}\,\phi\,(k+1,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1)\\
&\qquad + (-1)^{\alpha_1+\beta_1}\alpha_2\sqrt{\alpha+1}\,\phi\,(k+1,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2)\\
&\qquad + (1-\alpha_1)\sqrt{\alpha}\,\phi\,(k+1,l+1,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&\qquad + (1-\alpha_1)\sqrt{\alpha}\,k\,\phi\,(k,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&\qquad + (-1)^{\alpha_2}(1-\alpha_1)(1-\alpha_2)\sqrt{\alpha}\,\phi\,(k,l,\alpha_1+1,\beta_1,\alpha_2+1,\beta_2+1)\\
&\qquad + (1-\alpha_1)\alpha_2\sqrt{\alpha}\,\phi\,(k,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&L(V_-)\,\phi\,(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad = (1-\alpha_1)\sqrt{\alpha}\,\phi\,(k,l+1,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&\qquad + (1-\alpha_1)\sqrt{\alpha}\,k\,\phi\,(k-1,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&\qquad + (-1)^{\alpha_1+\beta_1+\alpha_2}\sqrt{\alpha+1}\,\phi\,(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1)\\
&\qquad + (-1)^{\alpha_1+\beta_1}\alpha_2\sqrt{\alpha+1}\,k\,\phi\,(k,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2)\\
&L(W_+)\,\phi\,(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad = (-1)^{\alpha_1}\sqrt{\alpha}\;\phi\,(k+1,l,\alpha_1,\beta_1+1,\alpha_2,\beta_2)\\
&\qquad + (-1)^{\alpha_1+\beta_1}(n-\alpha_1)(1-\alpha_2)\sqrt{\alpha+1}\,\phi\,(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
&\qquad + \alpha_1\sqrt{\alpha}\,\phi\,(k+1,l,\alpha_1-1,\beta_1,\alpha_2,\beta_2)\\
&\qquad + (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\sqrt{\alpha+1}\,\phi\,(k+1,l+1,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
&\qquad + (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\sqrt{\alpha+1}\,k\,\phi\,(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
&\qquad - (-1)^{\beta_2}(1-\alpha_1)(1-\alpha_2)\sqrt{\alpha+1}\,\phi\,(k,l,\alpha_1+1,\beta_1+1,\alpha_2+1,\beta_2)\\
&L(W_-)\,\phi\,(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad = -(-1)^{\alpha_1}\sqrt{\alpha}\;\phi\,(k,l,\alpha_1,\beta_1+1,\alpha_2,\beta_2)\\
&\qquad + (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\sqrt{\alpha+1}\,\phi\,(k,l+1,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
&\qquad + (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\sqrt{\alpha+1}\,k\,\phi\,(k-1,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
&\qquad - \alpha_1\sqrt{\alpha}\,\phi\,(k,l,\alpha_1-1,\beta_1,\alpha_2,\beta_2)
\end{aligned}
$$

From (9), it follows that the sum $(l+\beta_1+\beta_2)$ does not decrease under the action of the representation L and the subspace

$$V_m = \{\phi\,(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2) \in V \mid l+\beta_1+\beta_2 \geq m\} \tag{10}$$

is invariant, for which no invariant complementary subspace exists. Thus, the representation given by (9) on the space V is indecomposable.

The generalised Fock space is defined as a quotient space of V

$$Y = (V/J)\colon \{\phi(k, \alpha_1, \alpha_2) = \phi\,(k, 0, \alpha_1, 0, \alpha_2, 0) \bmod J \mid k \in Z^+, \alpha_1, \alpha_2 = 0, 1\} \tag{11}$$

where J is the left ideal generated by the elements $b - \lambda$, $a_1 - \eta_1$, and $a_2 - \eta_2$; λ is a complex number, and η_1 and η_2 are generators of the Grassmann algebra $\tilde{G}$. On this space, the representation (9) induces the new representation

$$\begin{aligned}
&L(Q_3)\,\phi\,(k, \alpha_1, \alpha_2)\\
&\quad = (-n + 2k + \alpha_1 + \alpha_2)\,\phi\,(k, \alpha_1, \alpha_2)\\
&\qquad + 2\lambda\,\phi\,(k+1, \alpha_1, \alpha_2) + (-1)^{\alpha_1}(1-\alpha_1)\eta_1\,\phi\,(k, \alpha_1 + 1, \alpha_2)\\
&\qquad + (-1)^{\alpha_2}(1-\alpha_2)\eta_2\,\phi\,(k, \alpha_1, \alpha_2 + 1)\\
&L(B)\,\phi\,(k, \alpha_1, \alpha_2)\\
&\quad = [(\alpha+1)n - \alpha_1 + \alpha_2]\,\phi\,(k, \alpha_1, \alpha_2)\\
&\qquad - (-1)^{\alpha_1}(1-\alpha_1)\eta_1\,\phi\,(k, \alpha_1+1, \alpha_2) + (-1)^{\alpha_2}\,(1-\alpha_2)\eta_2\,\phi\,(k, \alpha_1, \alpha_2+1)\\
&L(M)\,\phi\,(k, \alpha_1, \alpha_2)\\
&\quad = [(\alpha+1)n - k - \alpha_1]\,\phi\,(k, \alpha_1, \alpha_2)\\
&\qquad - (-1)^{\alpha_1}(1-\alpha_1)\eta_1\,\phi\,(k, \alpha_1+1, \alpha_2) - \lambda\,\phi\,(k+1, \alpha_1, \alpha_2)\\
&L(Q_+)\,\phi\,(k, \alpha_1, \alpha_2)\\
&\quad = (n - k - \alpha_1 - \alpha_2)\,\phi\,(k+1, \alpha_1, \alpha_2) - \lambda\,\phi\,(k+2, \alpha_1, \alpha_2)\\
&\qquad - (-1)^{\alpha_1}(1-\alpha_1)\eta_1\,\phi\,(k+1, \alpha_1+1, \alpha_2)\\
&\qquad - (-1)^{\alpha_2}(1-\alpha_2)\eta_2\,\phi\,(k+1, \alpha_1, \alpha_2+1)\\
&L(Q_-)\,\phi\,(k, \alpha_1, \alpha_2) = \lambda\,\phi\,(k, \alpha_1, \alpha_2) + k\,\phi\,(k-1, \alpha_1, \alpha_2) \qquad (12)\\
&L(V_+)\,\phi\,(k, \alpha_1, \alpha_2)\\
&\quad = -\sqrt{\alpha}n(1-\alpha_1)\,\phi\,(k, \alpha_1+1, \alpha_2)\\
&\qquad + (-1)^{\alpha_1+\alpha_2}\sqrt{\alpha+1}\;\eta_2\,\phi\,(k+1, \alpha_1, \alpha_2)\\
&\qquad + (-1)^{\alpha_1}\alpha_2\,\sqrt{\alpha+1}\,\phi\,(k+1, \alpha_1, \alpha_2-1)\\
&\qquad + (1-\alpha_1)\sqrt{\alpha}\,\lambda\,\phi\,(k+1, \alpha_1+1, \alpha_2) + (1-\alpha_1)\sqrt{\alpha}\,k\,\phi\,(k, \alpha_1+1, \alpha_2)\\
&\qquad + (-1)^{\alpha_2}(1-\alpha_1)(1-\alpha_2)\sqrt{\alpha}\;\eta_2\,\phi\,(k, \alpha_1+1, \alpha_2+1)\\
&\qquad + (1-\alpha_1)\,\alpha_2\sqrt{\alpha}\,\phi\,(k, \alpha_1+1, \alpha_2)
\end{aligned}$$

$$
\begin{aligned}
&L(V_-)\,\phi\,(k,\alpha_1,\alpha_2)\\
&\quad=(1-\alpha_1)\sqrt{\alpha}\,\lambda\,\phi\,(k,\alpha_1+1,\alpha_2)\\
&\qquad+(1-\alpha_1)\sqrt{\alpha}\,k\,\phi\,(k-1,\alpha_1+1,\alpha_2)+(-1)^{\alpha_1+\alpha_2}\sqrt{\alpha+1}\,\eta_2\,\phi\,(k,\alpha_1,\alpha_2)\\
&\qquad+(-1)^{\alpha_1}\alpha_2\sqrt{\alpha+1}\,k\,\phi\,(k,\alpha_1,\alpha_2-1)\\
&L(W_+)\,\phi\,(k,\alpha_1,\alpha_2)\\
&\quad=(-1)^{\alpha_1}(n-\alpha_1)\,(1-\alpha_2)\sqrt{\alpha+1}\,\phi\,(k,\alpha_1,\alpha_2+1)\\
&\qquad+(-1)^{\alpha_1}\sqrt{\alpha}\,\eta_1\,\phi\,(k+1,\alpha_1,\alpha_2)+\alpha_1\sqrt{\alpha}\,\phi\,(k+1,\alpha_1-1,\alpha_2)\\
&\qquad+(-1)^{\alpha_1}(1-\alpha_2)\sqrt{\alpha+1}\,\lambda\,\phi\,(k+1,\alpha_1,\alpha_2+1)\\
&\qquad-(-1)^{\alpha_1}(1-\alpha_2)\sqrt{\alpha+1}\,k\,\phi\,(k,\alpha_1,\alpha_2+1)\\
&\qquad-(1-\alpha_1)(1-\alpha_2)\sqrt{\alpha+1}\,\eta_1\,\phi\,(k,\alpha_1+1,\alpha_2+1)\\
&L(W_-)\,\phi\,(k,\alpha_1,\alpha_2)\\
&\quad=(-1)^{\alpha_1}(1-\alpha_2)\sqrt{\alpha+1}\,\lambda\,\phi\,(k,\alpha_1,\alpha_2+1)\\
&\qquad+(-1)^{\alpha_1}(1-\alpha_2)\sqrt{\alpha+1}\,k\,\phi\,(k-1,\alpha_1,\alpha_2+1)\\
&\qquad-(-1)^{\alpha_1}\sqrt{\alpha}\,\eta_1\,\phi\,(k,\alpha_1,\alpha_2)-\alpha_1\sqrt{\alpha}\,\phi\,(k,\alpha_1-1,\alpha_2)
\end{aligned}
$$

The representation given by (12) is an infinite-dimensional irreducible representation for the cases $\lambda \neq 0$, $\eta_1 \neq 0$, or $\eta_2 \neq 0$. When $\lambda = \eta_1 = \eta_2 = 0$, the representation (12) becomes

$$
\begin{aligned}
L(Q_3)\,\phi\,(k,\alpha_1,\alpha_2)&=(-n+2k+\alpha_1+\alpha_2)\,\phi\,(k,\alpha_1,\alpha_2)\\
L(B)\,\phi\,(k,\alpha_1,\alpha_2)&=[(\alpha+1)n-\alpha_1+\alpha_2]\,\phi\,(k,\alpha_1,\alpha_2)\\
L(M)\,\phi\,(k,\alpha_1,\alpha_2)&=[(\alpha+1)n-k-\alpha_1]\,\phi\,(k,\alpha_1,\alpha_2)\\
L(Q_+)\,\phi\,(k,\alpha_1,\alpha_2)&=(n-k-\alpha_1-\alpha_2)\,\phi\,(k+1,\alpha_1,\alpha_2)\\
L(Q_-)\,\phi\,(k,\alpha_1,\alpha_2)&=k\,\phi\,(k-1,\alpha_1,\alpha_2) \qquad (13)\\
L(V_+)\,\phi\,(k,\alpha_1,\alpha_2)&=(\alpha_2-n+k)(1-\alpha_1)\sqrt{\alpha}\,\phi\,(k,\alpha_1+1,\alpha_2)\\
&\quad+(-1)^{\alpha_1}\alpha_2\sqrt{\alpha+1}\,\phi\,(k+1,\alpha_1,\alpha_2-1)\\
L(V_-)\,\phi\,(k,\alpha_1,\alpha_2)&=(1-\alpha_1)\sqrt{\alpha}\,k\,\phi\,(k-1,\alpha_1+1,\alpha_2)\\
&\quad+(-1)^{\alpha_1}\alpha_2\sqrt{\alpha+1}\,k\,\phi\,(k,\alpha_1,\alpha_2-1)\\
L(W_+)\,\phi\,(k,\alpha_1,\alpha_2)&=(-1)^{\alpha_1}(n-k-\alpha_1)(1-\alpha_2)\sqrt{\alpha+1}\,\phi\,(k,\alpha_1,\alpha_2+1)\\
&\quad+\alpha_1\sqrt{\alpha}\,\phi\,(k+1,\alpha_1-1,\alpha_2)\\
L(W_-)\,\phi\,(k,\alpha_1,\alpha_2)&=(-1)^{\alpha_1}(1-\alpha_2)\sqrt{\alpha+1}\,k\,\phi\,(k-1,\alpha_1,\alpha_2+1)\\
&\quad-\alpha_1\sqrt{\alpha}\,\phi\,(k,\alpha_1-1,\alpha_2)
\end{aligned}
$$

We can easily see that the representation (13) is an infinite-dimensional irreducible representation when $n \notin Z^+$. Obviously, the invariant subspace exists when $n \in Z^+$,

$$Y(n)\colon \{\phi(k,\alpha_1,\alpha_2) \in Y \mid k+\alpha_1+\alpha_2 \leq n,\, k \in Z^+,\, \alpha_1,\alpha_2 = 0{,}1\} \tag{14}$$

$$\dim Y(n) = 4n \tag{15}$$

and there is no invariant complementary subspace. Thus, the representation (13) is indecomposable. Restricting the representation given by (13) to the invariant subspace $Y(n)$, we can obtain a finite-dimensional irreducible representation of the gl(2 | 1).

3. ONE-PARAMETER IRREDUCIBLE REPRESENTATIONS OF THE gl(2 | 1)

For the sake of simplicity, we redefine the basis of $Y(n)$ as

$$|j,m,\alpha_1,\alpha_2\rangle = \sqrt{\frac{(j-m)!(2j-\alpha_1-\alpha_2)!}{(j+m)!(j-m-\alpha_1)!(j-m-\alpha_2)!}}\,\phi(j+m,\alpha_1,\alpha_2) \tag{16}$$

where

$$j = \frac{1}{2}n = 0, \frac{1}{2}, 1\frac{3}{2}, 2, \ \ldots\ldots$$

$$\begin{aligned}
m &= -j, -j+1, \ldots, j, && \text{when } \alpha_1 = 0, \ \ \alpha_2 = 0\\
m &= -j, -j+1, \ldots, j-1, && \text{when } \alpha_1 = 0, \ \ \alpha_2 = 1\\
m &= -j, -j+1, \ldots, j-1, && \text{when } \alpha_1 = 1, \ \ \alpha_2 = 0\\
m &= -j, -j+1, \ldots, j-2, && \text{when } \alpha_1 = 1, \ \ \alpha_2 = 1.
\end{aligned}$$

The action of the generators of the gl(2 | 1) on the new basis vector is straightforwardly obtained with the help of Eqs. (13) and (16). One finds

$$\begin{aligned}
Q_3|j,m,\alpha_1,\alpha_2\rangle &= (2m+\alpha_1+\alpha_2)|j,m,\alpha_1,\alpha_2\rangle\\
B|j,m,\alpha_1,\alpha_2\rangle &= [(\alpha+1)2j-\alpha_1+\alpha_2]|j,m,\alpha_1,\alpha_2\rangle\\
M|j,m,\alpha_1,\alpha_2\rangle &= [(2\alpha+1)j-m-\alpha_1]|j,m,\alpha_1,\alpha_2\rangle
\end{aligned} \tag{17}$$

$$\begin{aligned}
&Q_+|j,m,\alpha_1,\alpha_2\rangle\\
&\quad = (j-m-\alpha_1-\alpha_2)\sqrt{\frac{(j+m+1)(j-m)}{(j-m-\alpha_1)(j-m-\alpha_2)}}|j,m+1,\alpha_1,\alpha_2\rangle
\end{aligned}$$

$$Q_-|j,m,\alpha_1,\alpha_2\rangle = \sqrt{\frac{(j+m)(j-m+1-\alpha_1)(j-m+1-\alpha_2)}{j-m+1}}|j,m-1,\alpha_1,\alpha_2\rangle$$

$$V_+|j,m,\alpha_1,\alpha_2\rangle = (1-\alpha_1)\sqrt{\alpha}(\alpha_2-j+m)\sqrt{\frac{2j-\alpha_1-\alpha_2}{j-m-\alpha_1}}|j,m+1,\alpha_1-1,\alpha_2\rangle + (-1)^{\alpha_1}\alpha_2\sqrt{\alpha+1}\sqrt{\frac{(j+m+1)(j-m)}{(j-m-\alpha_1)(2j-\alpha_1-\alpha_2+1)}} \times |j,m+1,\alpha_1,\alpha_2-1\rangle$$

$$V_-|j,m,\alpha_1,\alpha_2\rangle = (-1)^{\alpha_1}\alpha_2\sqrt{\alpha+1}\sqrt{\frac{j-m-\alpha_2+1}{2j-\alpha_1-\alpha_2+1}}\,|j,m,\alpha_1,\alpha_2-1\rangle + (1-\alpha_1)\sqrt{\alpha} \times \sqrt{\frac{(j+m)(j-m-\alpha_2+1)(2j-\alpha_1-\alpha_2)}{j-m+1}}\,|j,m-1,\alpha_1+1,\alpha_2\rangle$$

$$W_+|j,m,\alpha_1,\alpha_2\rangle = \alpha_1\sqrt{\alpha}\sqrt{\frac{(j+m+1)(j-m)}{(j-m-\alpha_2)(2j-\alpha_1-\alpha_2+1)}}\,|j,m+1,\alpha_1-1,\alpha_2\rangle + (-1)^{\alpha_1}(1-\alpha_2)\sqrt{\alpha+1}(j-m-\alpha_1)\sqrt{\frac{2j-\alpha_1-\alpha_2}{j-m-\alpha_2}}\,|j,m,\alpha_1,\alpha_2+1\rangle$$

$$W_-|j,m,\alpha_1,\alpha_2\rangle = -\alpha_1\sqrt{\alpha}\sqrt{\frac{j-m-\alpha_1+1}{2j-\alpha_1-\alpha_2+1}}|j,m,\alpha_1-1,\alpha_2\rangle + (-1)^{\alpha_1}(1-\alpha_2)\sqrt{\alpha+1} \times \sqrt{\frac{(j+m)(j-m-\alpha_1+1)(2j-\alpha_1-\alpha_2)}{j-m+1}}\,|j,m-1,\alpha_1,\alpha_2+1\rangle$$

where we restrict $|j, j+1, \alpha_1, \alpha_2\rangle = |j, -j-1, \alpha_1, \alpha_2\rangle = 0$

To illustrate the irreducibility of the gl(2 | 1) representation, we have a simple discussion. In the first place, the representation space $Y(2j)$ set up by all $|j, m, \alpha_1, \alpha_2\rangle$ marked with j is invariant under the action of the gl(2 | 1) generators. In the next place, there is no true subspace in the $Y(2j)$. It is clear from Eq. (17) that this representation is an $8j$-dimensional irreducible representation.

We have obtained one-parameter indecomposable and irreducible representations. All the finite-dimensional one-parameter irreducible representations of the gl(2 | 1) have been given on the subspace of the generalized Fock space.

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